## Vector Spaces and Subspaces

① Let $v = (3,4) \in V$. Then:

$1 \boxdot v = 1 \boxdot (3,4) = (-4,3) \neq (3,4)$. So VS 5 Fails. (On the exam, you don't have to quote the number of the axiom.) There are other VS axioms that fail here, but you only need to pick one.

② If $T$ is linear, $T: V \to W$, then $T(0_V) = 0_W$.

Here $T(0_V) = 0_V$, so $0_V \in W$.

Let $x, y \in W$. Then $T(x+y) = T(x) + T(y) = x+y$. So $x+y \in W$.

Let $\lambda \in F$ and $x \in W$. Then $T(\lambda x) = \lambda T(x) = \lambda x$. So $\lambda x \in W$.

③ Let $f_0$ refer to the zero vector in $V$. That is, $f_0(x) = 0$ for all $x \in \mathbb{R}$.

Then $f_0(1) = 0 = -f(2)$, so $f_0 \in V$.

Suppose $f, g \in V$. Then $(f+g)(1) = f(1) + g(1) = -f(2) - g(2) = -(f+g)(2)$.

Then $f+g \in V$.

Suppose $f \in V$ and $\lambda \in \mathbb{R}$. Then $(\lambda f)(1) = \lambda(f(1)) = \lambda(-f(2)) = -(\lambda f)(2)$.

So $\lambda f \in V$.

④ Not a subspace. To prepare for the exam, come up with at least three reasons why not.

⑤

$$W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \middle| \ a,b,c \in \mathbb{R} \right\} = \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \middle| \ a,b,c \in \mathbb{R} \right\}.$$

Then $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ spans $W_1$.

Set $a_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Then $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so all the $a_i = 0$.

Then our spanning set is independent, so a basis.

$\dim(W_1) = 3$.

$$W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \middle| \ a,b \in \mathbb{R} \right\} = \left\{ a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \middle| \ a,b \in \mathbb{R} \right\}$$

$= \operatorname{span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Setting $a_1\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + a_2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$,

we get $\begin{pmatrix} 0 & a_1 \\ -a_1 & a_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so $a_1 = a_2 = 0$.

$\dim(W_2) = 2$

For $W_1 \cap W_2$, since $a_{11} = 0$ in $W_2$ and $a_{11} = a_{22}$ in $W_1$, $a_{11} = a_{22} = 0$ in the intersection. $W_1$ has no restriction on $a_{12}$ and $a_{21}$, so only the restriction in $W_2$ applies. Then $W_1 \cap W_2 = \operatorname{span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$. $\dim(W_1 \cap W_2) = 1$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \in W_1 + W_2$.

Since $A$ is arbitrary, we have $M_{2\times2}(\mathbb{R}) \subseteq W_1 + W_2$.

Since $W_1 + W_2 \subseteq M_{2\times2}(\mathbb{R})$ by definition, we have

$W_1 + W_2 = M_{2\times2}(\mathbb{R})$. So $\dim(W_1 + W_2) = 4$. A

basis would be the standard basis.

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Spanning, Independence, Bases.

① If $a(u+v) + b(u+2v) = 0$, then
$(a+b)u + (a+2b)v = 0$. Since $\{u,v\}$ is independent,
$a+b=0$ and $a+2b=0$. So $a+b-(a+2b) = -b = 0$. Then $a=0$.
So $\{u+v, u+2v\}$ is independent. Since it's an independent set of size 2 in a space of dimension 2, it's a basis.

② ⓐ Let $a_1v_1 + a_2v_2 = 0$ be a non-trivial representation of zero.
Then $T(a_1v_1 + a_2v_2) = a_1T(v_1) + a_2T(v_2) = 0$. Since then $a_1, a_2$ are not both zero, this is a non-trivial representation as well.

ⓑ This is false. Come up with a counterexample.

③ Consider the set $S = \{v_1, v_2, \ldots, v_{n-1}\} =$
$\{(1,0,\ldots,-1), (0,1,\ldots,-1), (0,0,1,\ldots,-1), \ldots, (0,0,\ldots 0,1,-1)\}$.
Then $S$ spans $W$, as $\sum_{i=1}^{n-1} a_iv_i = (a_1, a_2, \ldots, a_{n-1}, -(a_1+a_2+\cdots+a_{n-1}))$
If $\sum_{i=1}^{n-1} a_iv_i = 0$, then $a_1 = a_2 = \cdots = a_{n-1} = 0$, via corresponding coordinates with $(0,0,\ldots,0)$.
So $S$ is a basis, and $\dim(W) = n-1$.

④ Since $W_i \neq V$, $\dim W_i \leq n-1$. This is because a subset of $V$ of dimension $n$ would be equal to $V$.
Suppose $\dim(W_1 \cap W_2) = n-1$ as well. Since $W_1 \cap W_2 \subseteq W_1$, we'd have $W_1 \cap W_2 = W_1$ in that case. For the same reason we'd have $W_1 \cap W_2 = W_2$. Then, if $\dim(W_1 \cap W_2) = n-1$, $W_1 = W_2$. So $\dim(W_1 \cap W_2)$ is at most $n-2$.

(5) Suppose $a_1(v_1+w)+a_2(v_2+w)+\cdots+a_n(v_n+w)=0$, and not all the $a_i$ are zero.

Then $a_1v_1+\cdots+a_nv_n+\left(\sum_{i=1}^{n}a_i\right)w=0$.

Since $\{v_1,\ldots,v_n\}$ is independent, $\sum_{i=1}^{n}a_i\neq 0$. Let $\lambda=\frac{1}{\sum a_i}$.

This $\lambda$ exists, because $\sum_{i=1}^{n}a_i$ is a non-zero element in a field.

Then $w=\frac{a_1}{\lambda}v_1+\cdots+\frac{a_n}{\lambda}v_n$. Hence $w\in\operatorname{span}\{v_1,v_2,\ldots,v_n\}$.

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Transformations

(1) If $R(T)=N(T)$, then rank + nullity of $T$ is an even number. But the Dimension theorem says rank + nullity $=\dim(V)$, which is odd. So not possible.

Note: If $\dim(V)$ is even, this is possible.
$T:\mathbb{R}^2\to\mathbb{R}^2$ by $T(a,b)=(b,0)$.

(2) $T$ can't be onto anyway. The $\operatorname{rank}(T)\le\dim(\mathbb{R}^5)$ by the dimension theorem. So it's not possible for $R(T)=P_5(\mathbb{R})$, since $\dim(P_5)=6$.

(3) (a) $T(\lambda A+B)=T\begin{pmatrix}\lambda a_{11}+b_{11} & \lambda a_{12}+b_{12}\\ \lambda a_{21}+b_{21} & \lambda a_{22}+b_{22}\end{pmatrix}=\left(\lambda a_{11}+b_{11}-(\lambda a_{22}+b_{22}),\,-2(\lambda a_{22}+b_{22})-\lambda a_{12}+b_{12}\right)$

$$=\left(\lambda(a_{11}-a_{22})+b_{11}-b_{22},\ \lambda(-2a_{22}-a_{12})+(-2b_{22}-b_{12})\right)$$
$$=\lambda(a_{11}-a_{22},-2a_{22}-a_{12})+(b_{11}-b_{22},-2b_{22}-b_{12})$$
$$=\lambda T(A)+T(B).$$

(b) $N(T)$ satisfies $a_{11}=a_{22}$, $a_{12}=-2a_{22}$.

$$N(T)=\left\{\begin{pmatrix}a_{11} & -2a_{11}\\ a_{21} & a_{11}\end{pmatrix}\,\middle|\, a_{ij}\in\mathbb{R}\right\}=\operatorname{span}\left\{\begin{pmatrix}1&-2\\0&1\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}.$$

These vectors are not multiples of each other, so independent.
Then a basis for $N(T)$ is $\left\{\begin{pmatrix}1&-2\\0&1\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}$.

$T(E^{11})=(1,0)$, $T(E^{12})=(0,-1)$. Since $\{(1,0),(0,-1)\}$ is independent, it must be a basis for $R(T)$, as $\dim V-\dim(N(T))=4-2=2$. So we know $\dim(R(T))=2$.

4. (a) $T(P_2(\mathbb{R}) \to P_3(\mathbb{R}))$ by $T(f) = f$. Clearly $N(T) = \{0\}$, so $T$ is one-to-one. But $T$ is not onto, as $x^3 \notin R(T)$.

(b) $T: \mathbb{R}^2 \to \mathbb{R}$ by $T(a,b) = a$.
Not one-to-one as $T(1,0) = T(1,1)$.
Onto, as for $x \in \mathbb{R}$, $T(x,0) = x$.

5. $$\begin{aligned} T(\lambda f+g) &= x((\lambda f+g)(x)) + \frac{d}{dx}(\lambda f+g)(x) \\ &= x((\lambda f)(x) + g(x)) + \frac{d}{dx}((\lambda f)(x)) + \frac{d}{dx}(g(x)) \\ &= \lambda x f(x) + x g(x) + \lambda f'(x) + g'(x) \\ &= \lambda x f(x) + \lambda f'(x) + x g(x) + g'(x) \\ &= \lambda T(f(x)) + T(g(x)). \end{aligned}$$